Eur. Phys. J. B **6**, 419–424 (1998)

THE EUROPEAN PHYSICAL JOURNAL B

Surface flows of granular mixtures: II. Segregation with grains of different size

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Received: 25 February 1998 / Received in final form and Accepted: 6 July 1998

Abstract. We present the generalization of a theoretical model for segregation of granular mixtures due to surface flows, published in J. Phys. I France **6**, 1295 (1996). Our generalized model is valid for grains differing by their size and/or their surface properties; in the present paper, we describe the case of two species with the same surface properties but two different sizes. The rolling stream is assumed to be homogeneous. Exchanges between the grains at rest and the rolling stream are modelized via binary collisions. The model predicts that during the filling of a two-dimensional silo, *continuous segregation* appears inside the static phase: small (respectively large) grains tend to stop uphill (respectively downhill), although both species remain present everywhere. This fits the observations when the size difference between the species is small. When the size difference is large, a different regime is observed. We argue that in this case, segregation occurs *directly inside* the rolling stream.

PACS. 47.55.Kf Multiphase and particle-laden flows – 83.10.Pp Particle dynamics – 83.70.Fn Granular solids

1 Introduction

Segregation is a phenomena commonly observed in granular materials that are poured, vibrated, or rotated [1]. Recently an experimental set up has focused much attention [2–4]: inside a two-dimensional "granular Hele-Shaw cell" (two vertical plates separated by a gap of approximately 5 mm), one pours a mixture of two granular species differing by their size and/or their surface properties (shape, roughness, sticking). The flowing particles stop, and progressively form a heap in the cell. *Segregation* is then observed in the following way: if the two species do not differ too much in size, a continuous segregation is obtained with more large or smooth grains at the bottom of the pile and more small or rough grains at the top, the transition zone being a few centimeters (the order of magnitude of the grain size is 0.5 mm). If the two species have a large difference in size, there are two cases: when the smaller grains are also the rougher, a complete segregation is obtained with all the large grains at the bottom, all the small ones at the top, and a transition zone between the two species of a few millimeters at maximum; when the larger grains are also the rougher, we obtain a spectacular effect called *stratification*: the particles deposit in alterning layers of different species, parallel to the sandpile surface.

The theoretical study of surface flows of a pure granular species has significantly progressed with the works of and Mehta and Barker [5], and Bouchaud, Cates, Prakash, and Edwards (BCRE) [6]. Recently in a first article of a series, Boutreux and de Gennes [7] generalized the BCRE equations to the case of a mixture of two species, in order to study the segregation in mixtures poured in two-dimensional silos. They proposed a theoretical formalism, and a model (called "minimal model") concentrating mainly on the case of grains with the same size but different surface properties. In this specific case, the minimal model was able to explain the observed continuous segregation, and predicted a power law behaviour of the concentrations. Károlyi *et al.* [8] have numerically simulated the situation described by Boutreux and de Gennes [7], using a granular media lattice gas model. The segregation obtained in the simulation was in very good agreement with the power law behaviour predicted in reference [7]. The simulation did not take into account a size difference between the two species (the minimal model of Boutreux *et al.* did not include such a difference). Very recently, Makse, Cizeau, and Stanley [9] modified the equations proposed by [7], to obtain a model for a mixture of grains with a large difference of sizes: they could explain the complete segregation by predicting an exponential behaviour of the concentrations, and they could successfully reproduce the mechanism leading to stratification as observed in the experiments [2].

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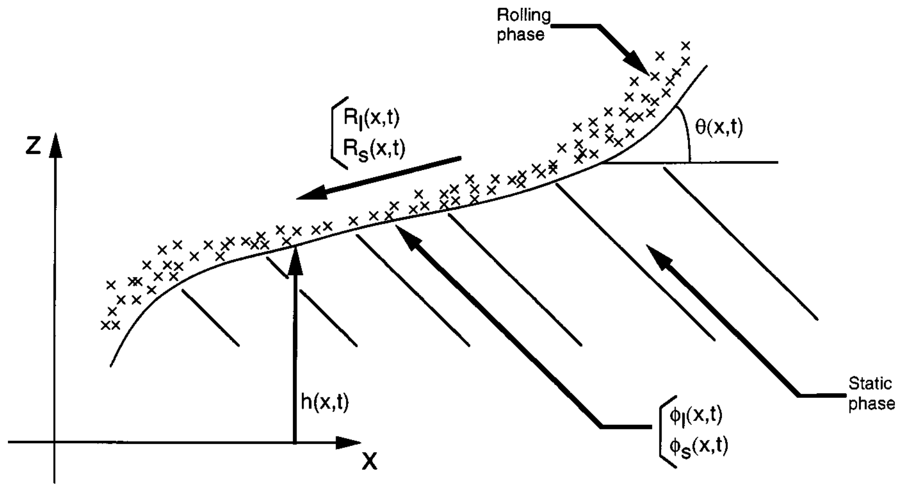

Fig. 1. Grains of the static phase are at rest. Grains of the rolling phase flow downhill, because of gravity. Exchanges between the two phases are due to collisions of rolling grains with static grains.

In the present paper, part II of the series started by Boutreux and de Gennes, we present a generalization of the minimal model proposed by Boutreux *et al.* [7]. This generalization is called "canonical model", and is valid when the two flowing species differ by their size and/or their surface properties. As we shall see, contrary to the model of Makse *et al.* [9], our generalization includes the situation where the species have a small difference of sizes. In the present article, we treat a particular case in order to explain the principles of our model: we describe the important case where the particles have different sizes but the same surface properties, and then the same angle of repose θ_r; we also assume here that the sizes of the two species are not too much different, and that the two densities are close. The general case of the canonical model that takes into account both the differences of size and of surface properties will be published soon [10]. The present paper is organized as follows: in the Section 2, we recall the theoretical formalism of Boutreux and de Gennes. We explain our microscopic description of the grain collisions, and derive a model for grains with identical surface properties in the Section 3. We then show, in the Section 4, that during the steady state filling of a two-dimensional silo, we predict continuous segregation, with a power law behaviour. We finally argue in the Section 5 that if the two granular species have a large difference of size, segregation happens directly inside the rolling phase; if we include this effect in the canonical model, we are able to explain the observed complete segregation and stratification. For this case, we compare our model and the model proposed by Makse *et al.* [9].

2 Model

Numerical simulations [11] have shown that in granular surface flows, there is a sharp distinction between a static phase where grains are at rest, and a thin rolling phase on top of the static phase. As proposed by Bouchaud *et al.* [6], this distinction is the starting point of the model (see Fig. 1). The angle $\theta(x,t)$ denotes the local slope of the interface, and $h(x,t)$ is the height of the static phase. We call $\phi_\alpha(x,t)$ the volume fractions of the two species of grains in the static phase just below the interface (here the index α denotes the grain species: "l" for large, and "s" for small); we have $\phi_l + \phi_s = 1$. The total thickness of the rolling phase is $R(x,t)$. We assume that the rolling phase is homogeneous in the vertical direction, *i.e.* there is no segregation already inside this phase. This is plausible if the particle sizes are not too much different; we come back to this point in the Section 5. We call $R_\alpha(x,t)$ the "equivalent thicknesses" of the two different species in the rolling phase: $R_\alpha(x,t)$ is equal to R multiplied by the local volume fraction of the α grains in the rolling phase ($R_l + R_s = R$).

As explained by Boutreux and de Gennes [7], the equation that describes the *exchanges* of grains between the two phases, due to collisions of rolling grains with static particles, is written:

$$\dot{h} = -(\dot{R}_l|_{coll} + \dot{R}_s|_{coll}), \tag{1a}$$

where the dot denotes a time derivative, and where $\dot{R}_\alpha|_{coll}$ describes the exchange of the α grains from the static phase to the rolling phase. Equation (1a) can also be written for a single species:

$$\phi_\alpha \dot{h} = -\dot{R}_\alpha|_{coll}. \tag{1b}$$

For the rolling phase, the evolution equation for each species is written:

$$\dot{R}_\alpha = v\frac{\partial R_\alpha}{\partial x} + \dot{R}_\alpha|_{coll}, \tag{1c}$$

where v is the speed of the rolling grains, convected downhill because of their weight. The value of the speed v is determined by a balance between the gravitational force, and the energy losses in collisions and friction between

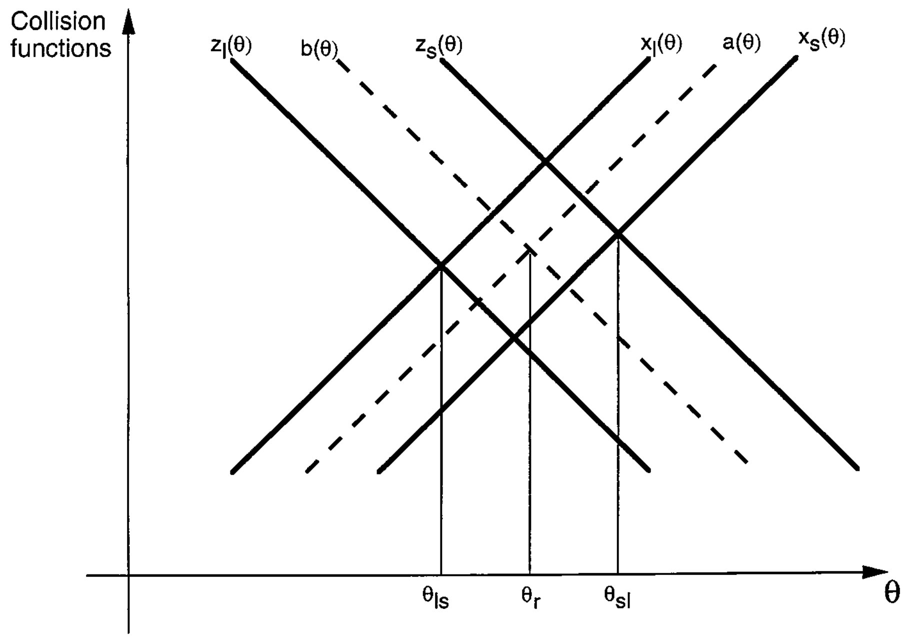

Fig. 2. The six different collision functions. The size difference between the two species allows to compare the functions. As a consequence, we obtain $\theta_{ls} < \theta_r < \theta_{sl}$.

grains; the speed v is then an increasing function of the restitution coefficient of the energy during collisions. Note that experiments also show that the value of v is larger for grains located in the upper part of the rolling phase [12]; in order to simplify our model, we will neglect this effect. Moreover, the value of v should depend on the slope $\theta(x,t)$. Since in practice θ is nearly constant (its variations are smaller than $\pm$ 5°), the speed v can be taken as constant.

3 Generalized angles of repose

We still need a microscopic model of the grain collisions, in order to express the exchange terms $\dot{R}_\alpha|_{coll}$ as functions of θ and R_α. We calculate these terms in the first order approximation, by considering only the *binary collisions* between one rolling grain, and one grain at rest on top of the static phase. When the rolling grain is large, there exist four types of collisions: (1) auto-amplification: another l grain starts to roll. This collision contributes a term $a_l(\theta)\phi_l R_l$ to $\dot{R}_l|_{coll}$. The term is proportional to R_l because the rolling phase being thin, all rolling grains interact with the static phase. The collision function a_l has the dimensions of a frequency; as a first approximation, it only depends on θ. (2) Cross-amplification: an s grain starts to move. It contributes a term $x_l(\theta)\phi_s R_l$ to $\dot{R}_s|_{coll}$. (3) Auto-capture: the l rolling grain is captured by an l static grain. It contributes a term $-b_l(\theta)\phi_l R_l$ to $\dot{R}_l|_{coll}$. (4) Cross-capture: the l rolling grain is captured by an s static grain. It contributes a term $-z_l(\theta)\phi_s R_l$ to $\dot{R}_l|_{coll}$. This cross-capture was not considered in the minimal model [7]; it plays an important role when the grains have different sizes. Four similar binary collisions happen when the rolling grain belongs to the s species. We call the corresponding collision functions $a_s(\theta)$, $x_s(\theta)$, $b_s(\theta)$, and $z_s(\theta)$. We therefore consider eight positive collision functions. Since increasing the slope favours rolling, a_α and x_α (amplification) are increasing functions of θ, and b_α and z_α (capture) are decreasing functions of θ. The size difference between the two species allows to compare the functions. A large grain sets more easily a small grain into move than the reverse: $x_l(\theta) > x_s(\theta)$ (see Fig. 2). A small grain is more easily captured by a large grain than the reverse: $z_s(\theta) > z_l(\theta)$. In the first order approximation, the probability that an auto-capture or an auto-amplification happens does not depend on the grain size: $a_l(\theta) \simeq a_s(\theta) \equiv a(\theta)$, and $b_l(\theta) \simeq b_s(\theta) \equiv b(\theta)$. The size difference also yields:

$$x_l > a > x_s, \quad z_s > b > z_l. \tag{2}$$

In the present paper, we shall concentrate on the simplest case among all the possible ones in the frame of the model. Therefore, we take the simplest relationships consistent with equation (2): $a = (x_l + x_s)/2$, and $b = (z_l + z_s)/2$, as shown in Figure 2.

In order to simplify the exchange terms $\dot{R}_\alpha|_{coll}$, it is useful to consider E_l, the exchange from the static phase to the rolling phase, due to collisions between l rolling grains and both kinds of static grains. By considering the involved binary collisions, we obtain the expression of E_l: $E_l = [(a-b)\phi_l + (x_l - z_l)\phi_s]R_l$. In a model of surface flows for a pure species, we would have: $E = (a-b)R$; at the angle θ for which $a = b$, there would be no exchange ($E = 0$): this angle is the repose angle θ_r of the pure grains. Moreover, in the case of a mixture, the angle for which $x_l = z_l$ will be called θ_{ls}. Let us now simplify the expression of E_l, by linearizing the collision functions with respect to θ. This approximation is possible for two different reasons: firstly, θ is always close to the angle of repose θ_r, as shown by the experiments made in cells [2–4]; moreover, we consider here the case of two granular species with similar sizes, as discussed in the last section of the paper. In order to consider the simplest case, we assume that the derivatives of the collision functions have the same order of magnitude (as $\partial_\theta a \simeq \partial_\theta x_l$). We finally obtain:

$$E_l = \gamma[\theta - \theta_l(\phi_s)]R_l, \tag{3}$$

where:

$$\theta_l(\phi_s) = \theta_r + (\theta_{ls} - \theta_r)\phi_s. \tag{4a}$$

The constant γ is given by: $\gamma = \partial_\theta a - \partial_\theta b$. It has the dimensions of a frequency; the larger the value of γ, the more frequent the exchanges between phases. A dimensional analysis shows that $\gamma \simeq v/d$, where d is the typical size of grains. Figure 3 represents the angle $\theta_l(\phi_s)$ given by equation (4a). Expression (3) shows that $\theta_l(\phi_s)$ is a cross-over angle: equation (3) describes capture of l rolling grains ($E_l < 0$) when $\theta < \theta_l$, and amplification ($E_l > 0$) when $\theta > \theta_l$. The angle $\theta_l(\phi_s)$ plays for a mixture of grains the role of the constant θ_r for a pure species. We call θ_l the *generalized angle of repose* of the l grains. This angle was introduced in [9], where its expression was postulated and not derived by considering microscopic collisions. When no s grain is present on top of the static phase ($\phi_s = 0$), the generalized angle of repose θ_l is equal to the angle of repose θ_r of the pure l species. If ϕ_s increases, the l rolling grains amplify more easily the static grains, and stop less easily in collisions with static grains: E_l increases *i.e.* the generalized angle of repose $\theta_l(\phi_s)$ decreases, as shown in Figure 3. With similar approximations, we get a simplified expression of the amplification of the rolling phase E_s due to collisions with s rolling grains: $E_s = \gamma[\theta - \theta_s(\phi_l)]R_s$, where:

$$\theta_s(\phi_l) = \theta_r + (\theta_{sl} - \theta_r)\phi_l. \tag{4b}$$

The generalized angle of repose $\theta_s(\phi_l)$ of the s grains is an increasing function of ϕ_l, as represented in Figure 3. The generalized angles θ_l and θ_s have a key role in the model. The angle of repose θ_r of both pure species is equal, but due to the size difference between the particles we have $\theta_{ls} < \theta_r < \theta_{sl}$, and $\theta_l(\phi_s) < \theta_s(\phi_l)$ for any value of ϕ_α: the l rolling species amplifies more easily the rolling phase than the s rolling species. This behaviour difference can be quantify through $\psi \equiv \theta_s - \theta_l$. The larger the size difference between the two species, the larger the value of ψ. Due to the approximations we made, here ψ is a constant independent of ϕ_α.

It is now possible to write simpler expressions for the exchange terms $\dot{R}_\alpha|_{coll}$. Let us define the "collision matrix" $\hat{M}$ by [7]: $\begin{pmatrix} \dot{R}_s|_{coll} \\ \dot{R}_l|_{coll} \end{pmatrix} = \hat{M} \begin{pmatrix} R_s \\ R_l \end{pmatrix}$. The previous calculations of E_α yield:

$$\hat{M} = \begin{pmatrix} \gamma[\theta - \theta_s(\phi_l)] - x_s(\theta)\phi_l & x_l(\theta)\phi_s \\ x_s(\theta)\phi_l & \gamma[\theta - \theta_l(\phi_s)] - x_l(\theta)\phi_s \end{pmatrix}, \tag{5a}$$

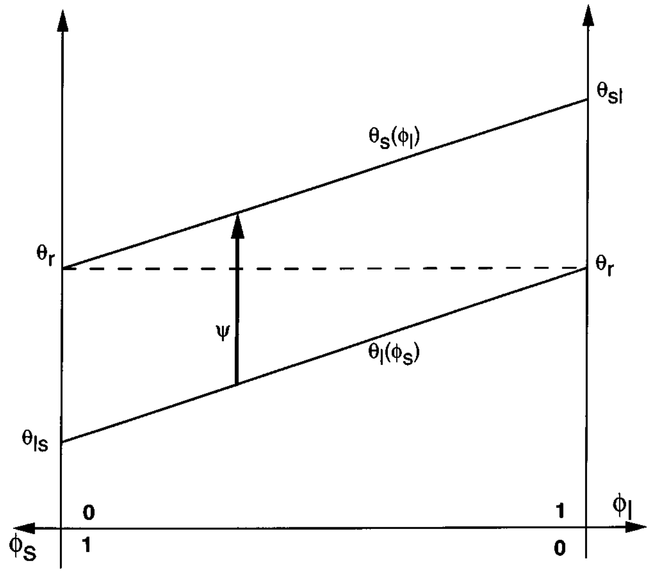

Fig. 3. The two generalized angles of repose: $\theta_l(\phi_s)$ for the large grains, and $\theta_s(\phi_l)$ for the small ones. Since the two species have the same surface properties, they have the same angle of repose θ_r. But due to the size difference, we obtain $\theta_l < \theta_s$ for any value of ϕ_α. Here, the difference $\psi \equiv \theta_s - \theta_l$ is constant.

where the cross-amplification functions can be written in the following way

$$x_s(\theta) = \frac{\gamma}{2}(\theta - \theta_r) + x_0, \quad x_l(\theta) = x_s(\theta) + \Delta x. \tag{5b}$$

The constants x_0 and Δx are two positive parameters of the model. As we saw (Eq. (2)), the difference $\Delta x = x_l - x_s$ is due to the size difference between the two species; the larger this size difference, the larger Δx.

4 Segregation in steady state

In order to describe the segregation between the two species, let us assume that we pour (at $x = L$) a constant flux of a given mixture, into the 2D cell located at $0 < x < L$ (Sect. 1); the interface rises uniformly at the constant speed $\dot{h}$. Some results, found for this situation with the minimal model [7], can be generalized to the case of a mixture of grains differing in their size. Equations (1c, 1a) imply that the total height $R(x)$ of the rolling phase decreases linearly with respect to the distance to the pouring point ($x = L$): $R(x) = x\dot{h}/v$. The total exchange between the two phases $\dot{R}_l|_{coll} + \dot{R}_s|_{coll}$ can be calculated, by using the expression of the collision matrix (5a); equation (1a) then yields $\gamma(\theta - \theta_l)R_l + \gamma(\theta - \theta_s)R_s = -\dot{h}$. In this expression the right hand side, being much smaller than the terms of the left hand side, can be neglected. We then get:

$$\theta = \theta_l(\phi_s)\frac{R_l}{R} + \theta_s(\phi_l)\frac{R_s}{R} \cdot \tag{6a}$$

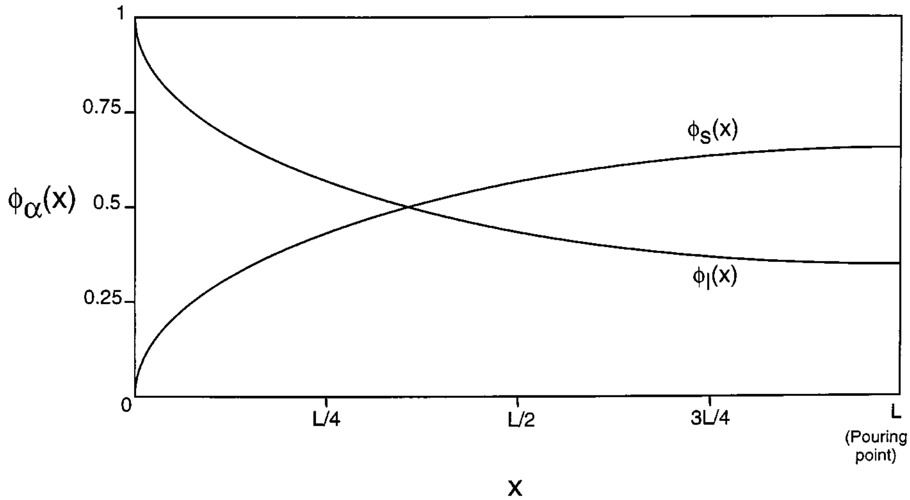

Fig. 4. The two volume fractions $\phi_\alpha(x)$ in the static phase, calculated numerically, for the steady state solution when a mixture (50% in volume for each species) is poured in a 2D cell. The model predicts continuous segregation inside the static phase. The numerical calculation is made with $\theta_r = 40°$, $\psi = 10°$, $x_0 = 0.3\gamma$, and $\Delta x = 0.1\gamma$.

In particular, in the steady state the value of the slope θ is always contained between the two generalized angles of repose θ_l and θ_s. By combining equations (1b, 5a, 6a), we obtain a relationship between R_α/R and the volume fractions in the static phase ϕ_α:

$$\phi_s = \left(1 + \xi \frac{R_l}{x_l R_l + x_s R_s}\right) \frac{R_s}{R} \tag{6b}$$

$$\phi_l = \left(1 - \xi \frac{R_s}{x_l R_l + x_s R_s}\right) \frac{R_l}{R},$$

where $\xi \equiv \gamma\psi - \Delta x$ is a constant in our model. Equation (6b) shows that *segregation* appears: the grains of the species α, for which $\phi_\alpha > R_\alpha/R$, stop more easily than the grains of the other species ($\phi_\beta < R_\beta/R$). Experiments [2,3] show that smaller grains stop first. Since R_α and x_α are always positive, this implies that in our model ξ must be positive, *i.e.* $\gamma\psi > \Delta x$. Returning to the definition of the generalized angles of repose, one shows that this condition is equivalent to $z_s(\theta) + x_s(\theta) > z_l(\theta) + x_l(\theta)$: cross-collisions involving small rolling grains are more frequent than those involving large rolling grains.

In the model, segregation is an increasing function of the parameter ξ: the larger the values of ψ and γ or the smaller the value of Δx, the stronger the segregation, *i.e.* the larger the size difference between the species. At the lower end of the slope ($x \ll L$), we have a complete purification of both species due to segregation: $R_s(x)/R$ and $\phi_s(x)$ tend to zero. In that region, it is possible to quantify more precisely segregation, by doing a power law development of these two quantities. We obtain that they vary as x^{ξ/x_0}, where $x_0 = x_s(\theta = \theta_r)$ as defined by equation (5b). Note that the exponent of this power law depends on the coefficients of the collision matrix. Moreover Figure 4 shows the volume fractions $\phi_\alpha(x)$ for $0 < x < L$, calculated numerically [10] in the case where the poured granular material is obtained by mixing the same volume of each species: $R_l = R_s$ at $x = L$. For this simulation, we chose $\theta_r = 40°$, $\psi = 10°$, $x_0 = 0.3\gamma$, and $\Delta x = 0.1\gamma$; experiments [3] show that this corresponds to a size ratio between the two species equal to approximately 1.2 (see Sect. 5). Note that segregation clearly appears at $x = L$, where $R_l = R_s$ but $\phi_l < \phi_s$. Figure 4 shows that our model predicts a continuous segregation, and not a complete one: ϕ_s does not fall rapidly at $x = L/2$, but progressively decreases as x decreases. The functions $R_\alpha(x)/R$ and $\theta(x)$ also vary progressively.

5 Discussion

We have presented a generalization of the model for surface flows of granular mixtures, proposed by Boutreux and de Gennes [7]. This generalization allows to describe mixtures of grains differing by their sizes. Let us call "ρ" the ratio of the size of the large particles divided by the size of the small ones. In this paper, we have assumed that ρ was close to one, *i.e.* that the sizes of the two granular species were not too much different. In this case, the rolling phase remains homogeneous and the collision functions are smooth functions of the slope θ. Hence we could linearized the collision functions with respect to θ. We then showed that our canonical model predicted continuous segregation. When $\rho < 1.5$ (approximately), this continuous segregation is observed in experiments [3], in qualitative agreement with our model. Quantitatively, the model predicts a power law behaviour for the concentrations; this prediction could be tested by experiments. Note that in the present paper, we restricted our scope to the case where grains differ only by their size. The general case of the canonical model, where the grain species differ by their size and/or their surface properties, will be published soon [10] (part III of this series).

When the two species have a large difference of size, *i.e.* $\rho > 1.5$ (approximately), experiments show that the continuous segregation is replaced by a complete one, or by stratification. Indeed, a new phenomena must appear inside the rolling phase, called interparticle "percolation" [13]. This is a gravity-induced, size-dependent, void-filling mechanism. The volume ratio and the mass ratio between the two species being large, the small particles fall through the gaps in between the large grains, and reach the lower part of the rolling phase. The large particles stay in the upper part of the phase, on top of the small grains. Hence already *inside the rolling phase*, we expect some segregation. The large rolling particles can not interact any more with the static phase, due to a screen effect produced by the small rolling particles. We must include this phenomena in our model of surface flows. When percolation happens, the collision functions of the large grains must be equal to zero, and the linear development we made in the present paper is not valid any more for these particles. Then the collision matrix must be written by using the collision functions; all the functions corresponding to collisions due to a large rolling grain must be set to zero. A numerical simulation of this modified canonical model has been done; a detailed description of our results will be published soon [10]. The simulation shows that we obtain either complete segregation or stratification, as observed in the experiments [2–4].

This simulation is also consistent with the results published by Makse, Cizeau, and Stanley [9], who already described theoretically the complete segregation and the stratification. Makse *et al.* [9] proposed a different model for surface flows of a mixture of two granular species. They use the theoretical formalism of Boutreux and de Gennes [7], but do not do the linear development of the collision functions. Instead, they propose some particular expressions for these functions, justified by physical arguments. Each function is equal to zero when the angle θ is larger or smaller than a given value. By allowing that the functions are set to zero, their model should implicitly include the percolation effect. Indeed, they predict either stratification or complete segregation, with an exponential behaviour. Then, the model of Makse *et al.* may be well adapted to describe experiments where the two grain species have a large difference of sizes. In contrast, our canonical model (with a linear development of the collision functions) describes mostly the case where the species have a small difference of sizes.

We thank P.-G. de Gennes, co-author of the first article of this series, for his very precious help. This work also benefited from many stimulating discussions with J.-P. Bouchaud, C. Gay, Y. Grasselli, H. J. Herrmann, H. A. Makse, J. Rajchenbach, and E. Raphaël.

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